

A PLASTICITY PRINCIPLE OF CONVEX QUADRILATERALS ON A CONVEX SURFACE OF BOUNDED SPECIFIC CURVATURE

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ABSTRACT. We derive the plasticity equations for convex quadrilaterals on a complete convex surface with bounded specific curvature and prove a plasticity principle which states that: *Given four shortest arcs which meet at the weighted Fermat-Torricelli point their endpoints form a convex quadrilateral and the weighted Fermat-Torricelli point belongs to the interior of this convex quadrilateral, an increase of the weight corresponding to a shortest arc causes a decrease of the two weights that correspond to the two neighboring shortest arcs and an increase of the weight corresponding to the opposite shortest arc* by solving the inverse weighted Fermat-Torricelli problem for quadrilaterals on a convex surface of bounded specific curvature. Furthermore, we show a connection between the plasticity of convex quadrilaterals on a complete convex surface with bounded specific curvature with the plasticity of some generalized convex quadrilaterals on a manifold which is certainly composed by triangles. We also study some cases of symmetrization of weighted convex quadrilaterals by introducing a new symmetrization technique which transforms some classes of weighted geodesic convex quadrilaterals on a convex surface to parallelograms in the tangent plane at the weighted Fermat-Torricelli point of the corresponding quadrilateral.

1. INTRODUCTION

We start with the definitions of the specific curvature of a domain W and the Gaussian curvature at a point P on a convex surface which is considered as the whole boundary of a convex body in \mathbb{R}^3 , and mention some fundamental results on a complete convex surface of bounded specific curvature from the book of A.D. Alexandrov (see [1]).

Definition 1 (see [1, pp. 365–366]). Any domain W on a convex surface has some curvature $\omega(W) = \iint_W K(P) dS$, where K is the Gaussian curvature at the point P and $S(W)$ is the corresponding area. The ratio $\frac{\omega(W)}{S(W)}$ is called the *specific curvature of the domain W* , and it is denoted by $\kappa(W)$. A convex surface has *Gaussian curvature equal to K at a point P* if the limit of the specific curvature of the domain tends to the limit K whenever this domain shrinks to the point P .

We recall the following fundamental results which have been formulated and proved by A.D. Alexandrov (see [1, pp. 365, 377: Theorems 2, 3, p. 7, 64–66: Theorem 3, p. 377: footnote 9]):

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(1) If the specific curvature $\kappa(W)$ of every domain W lying in some neighborhood of a point P on a convex surface does not exceed some positive number K then there exists $r_0 > 0$ such that it is possible to draw a shortest arc of length at least r_0 from the point P in each direction.

(2) Suppose that a *specific curvature* κ of a convex surface M is *bounded* on a neighborhood W , i.e., $K_1 \leq \kappa(W) \leq K_2$. Introduce the polar geodesic coordinates on the sphere S_{K_i} ($i = 1, 2$), and construct a mapping such that to each point of the neighborhood W with coordinates r, φ , correspond points of the spheres S_{K_i} for $i = 1, 2$, with the same coordinates. If, to a curve L on M , this mapping puts in correspondence the curves L_1 and L_2 on the spheres S_{K_1} and S_{K_2} , then the lengths of these curves are related by the inequalities

$$S(L_1) \geq S(L) \geq S(L_2).$$

(3) If the specific curvature $\kappa(W)$ is $\geq K$ ($\leq K$) in a triangle $\triangle ABC$, then the angles of $\triangle ABC$ are no less (or greater) than the corresponding angles of the triangle $\triangle ABC$ on the K -plane (see for the definition of K -plane in Section 3, p. 9). For $K > 0$, we consider the perimeter of the triangle to be lower than $\frac{2\pi}{\sqrt{K}}$.

(4) Each two points of a complete convex surface can be connected by a shortest arc.

(5) If the specific curvature $\kappa(W)$ is $\leq K$ on a complete convex surface, then each arc of a geodesic of length at most $\frac{\pi}{\sqrt{K}}$ is a shortest arc in general but not compared with close lines.

We consider M to be a complete convex surface with bounded specific curvature $K_1 < \kappa(W) < K_2$. The length of each shortest arc on W is greater than $\frac{\pi}{\sqrt{K_2}}$ and smaller than $\frac{\pi}{\sqrt{K_1}}$.

We state the weighted Fermat-Torricelli (w. F-T) problem on M for quadrilaterals.

Problem 1. Let $ABCD \subset M$ be a convex quadrilateral whose perimeter is less than $\frac{2\pi}{\sqrt{K_1}}$. Suppose that a positive number (weight) w_R , corresponds to the vertex $R \in \{A, B, C, D\}$. Find the w. F-T point P_F such that

$$f(P_F) = w_A l_A + w_B l_B + w_C l_C + w_D l_D \rightarrow \min \quad (1.1)$$

where l_R is the length of the shortest arc from the w. F-T point P_F to the vertex $R \in \{A, B, C, D\}$ (Fig. 1).

In the paper we provide the plasticity equations for convex quadrilaterals on a complete convex surface with bounded specific curvature M (Theorem 1 and Corollary 1) by applying a method of differentiation for shortest arcs which has been introduced for the differentiation of the length of geodesics in [11], [10] which generalize the plasticity equations introduced in [9] on the two dimensional K -plane (Two dimensional sphere, hyperbolic plane and Euclidean plane) and we prove the plasticity property of convex quadrilaterals on M which was numerically verified for convex quadrilaterals on the two dimensional K -plane in [9] without giving a proof.

The main result of the paper is the plasticity principle of convex quadrilaterals on M (Theorem 2 in Section 2) which states that:

Given four shortest arcs which meet at the weighted Fermat-Torricelli point and their endpoints form a convex quadrilateral and the weighted Fermat-Torricelli

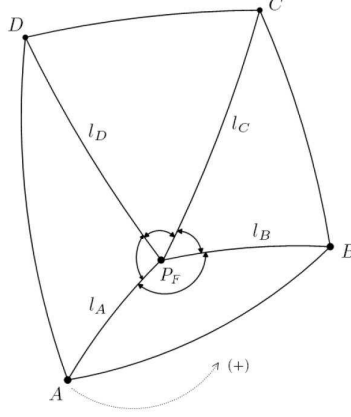


FIGURE 1.

point belongs to the interior of this convex quadrilateral, an increase of the weight corresponding to a shortest arc causes a decrease of the two weights that correspond to the two neighboring shortest arcs and an increase of the weight corresponding to the opposite shortest arc.

Using comparison geometry, we derive the plasticity equations for convex quadrilaterals for two cases of manifolds with metric of positive curvature which has been obtained by the gluing theorem of A.D. Alexandrov (Corollary 2 in Section 3). These results provide a generalization of the evolution of convex quadrilaterals which has been introduced in [9] on surfaces of constant Gaussian curvature.

2. THE 4-INVERSE WEIGHTED FERMAT-TORRICELLI PROBLEM ON A COMPLETE CONVEX SURFACE WITH BOUNDED SPECIFIC CURVATURE

Problem 2. *Given a point P which belongs to the interior of $ABCD$ on M , does there exist a unique set of positive weights w_i , such that*

$$w_A + w_B + w_C + w_D = c = \text{const},$$

for which P minimizes

$$f(P) = w_A l_A + w_B l_B + w_C l_C + w_D l_D$$

where l_R is the length of the shortest arc from P to the vertex $R \in \{A, B, C, D\}$.

This is the 4-inverse w. F-T problem on M (Fig. 1).

For $w_D = 0$, we derive the 3-inverse w. F-T problem which has been introduced and solved in \mathbb{R}^2 by S. Gueron and R. Tessler in [5] and it has been solved by [8] on a surface of constant Gaussian Curvature and generalized in [10], [11] on a C^2 surface. The solution of the 3-inverse weighted Fermat-Torricelli problem gives a positive answer on this problem on a C^2 surface. It is called the 3-inverse normalized weighted Fermat-Torricelli problem for $c = 1$ (see [5], page 449). We show that a solution of the 4-inverse weighted Fermat-Torricelli problem (Corollary 1) gives a negative answer to this problem because three weights depend on one variable weight (Theorem 1).

An application of the solution of the 4-inverse w. F-T problem is the derivation of the evolution of convex quadrilaterals on M . The evolution of convex quadrilaterals is given by the invariance of the weighted Fermat-Torricelli point for given convex quadrilaterals (geometric plasticity) and the plasticity of convex quadrilaterals which are convex quadrilaterals which satisfy Theorem 1 (dynamic plasticity). We note that the geometric plasticity was known to Viviani for the case of n given points in \mathbb{R}^2 .

Theorem 1. *Consider the 4-inverse w. F-T problem on a complete convex surface of bounded specific curvature M in \mathbb{R}^3 . The plasticity of convex quadrilaterals on M is given by the following three equations:*

$$\left(\frac{w_B}{w_A}\right)_{ABCD} = \left(\frac{w_B}{w_A}\right)_{ABC} \left[1 - \left(\frac{w_D}{w_A}\right)_{ABCD} \left(\frac{w_A}{w_D}\right)_{ACD}\right], \quad (2.1)$$

$$\left(\frac{w_C}{w_A}\right)_{ABCD} = \left(\frac{w_C}{w_A}\right)_{ABC} \left[1 - \left(\frac{w_D}{w_A}\right)_{ABCD} \left(\frac{w_A}{w_D}\right)_{ABD}\right], \quad (2.2)$$

and

$$(w_A)_{ABCD} + (w_B)_{ABCD} + (w_C)_{ABCD} + (w_D)_{ABCD} = \text{const.} \quad (2.3)$$

The weight $(w_R)_{ABCD}$ corresponds to the vertex R that lie on the shortest arc $P_F R$, for $R \in \{A, B, C, D\}$ and the weight $(w_S)_{SLN}$ corresponds to the vertex S that lie on the shortest arc $P_F S$ regarding the triangle $\triangle SLN$, for $S, L, N \in \{A, B, C, D\}$ and $S \neq L \neq N$.

Proof of Theorem 1: Firstly, we assume that we choose four initial (given) values $(w_R)_{ABCD}(0)$ concerning the weights $(w_R)_{ABCD}$ for $R \in \{A, B, C, D\}$ such that the w. F-T point P_F exists and it is located at the interior of $ABCD$. Differentiation of the objective function with respect to a specific arc length, yields the system

$$\sum_{Q \in \{A, B, C, D\}} w_Q \cos(\angle Q P_F R) = 0 \quad \text{for } R \in \{A, B, C, D\}.$$

Without loss of generality, we choose $Q \in \{A, B, C, D\}$. We suppose that the point Q , the arc length parameterized curve $c(t)$ and the point P_F that lie in $c(t)$ are given in the neighborhood W_{P_F} .

We apply the following general result given in [4, Corollary 4.5.7, p. 125, Remark 4.5.8, p. 126 and Theorem 4.5.6, p. 123] that deals with a rule for differentiating the length of a continuous family of shortest arcs connecting Q to points $c(t)$, for $Q \in \{A, B, C, D\}$ and the existence of the right derivative $\frac{d}{dt} l_Q$, although the shortest arcs from Q to $c(t)$ may not be unique on a non-negatively curved complete locally compact space X (M is a subset of X):

Let $c(t) : [0, T] \rightarrow M$ be arc length parameterized geodesic, Q a fixed point in M , $Q \neq c(0)$. Then the function $t \rightarrow l_Q(t) = |Qc(t)|$ has the right derivative and

$$\lim_{t \rightarrow +0} \frac{l_Q(t) - l_Q(0)}{t} = \cos(\pi - \varphi'_{c, \min}(t)) = \cos(\pi - \varphi'_c(t)), \quad (2.4)$$

where $\varphi'_{c, \min}(t)$ is the infimum (minimum) of angles between $c(t)$ and the shortest arcs connecting $c(0)$ to Q and takes a unique value $\varphi'_c(t)$ on M .

We choose the parametrization (a similar technique has been used in [11], [10] for the parametrization of length of geodesics):

$$l_A(t) = t, \quad (2.5)$$

We assume that the shortest arcs l_B, l_C, l_D , can be expressed as functions of l_A :

$$l_B = l_B(l_A), \quad l_C = l_C(l_A), \quad l_D = l_D(l_A). \quad (2.6)$$

From (2.6) and (1.1) the following equation is obtained:

$$w_A l_A + w_B l_B(l_A) + w_C l_C(l_A) + w_D l_D(l_A) = \min \quad (2.7)$$

Taking into account that the point P_F is the w. F-T point of $ABCD$ we derive that the right derivative of (2.7) with respect to the variable l_A and considering (2.5), we get

$$w_A + w_B \lim_{t \rightarrow +0} \frac{l_B(t) - l_B(0)}{l_A(t)} + w_C \lim_{t \rightarrow +0} \frac{l_C(t) - l_C(0)}{l_A(t)} + w_D \lim_{t \rightarrow +0} \frac{l_D(t) - l_D(0)}{l_A(t)} = 0. \quad (2.8)$$

From (2.4) and (2.5), for $Q = B$, we get

$$\lim_{t \rightarrow +0} \frac{l_B(t) - l_B(0)}{l_A(t)} = \cos(\angle AP_F B). \quad (2.9)$$

Let the point C , the length parameterized curve $c(t)$ and the point P_F that belongs in $c(t)$ be given in the neighborhood W_{P_F} . Taking into account (2.4) and (2.5), for $Q = C$, we have ([4, Corollary 4.5.7, p. 125, Remark 4.5.8, p. 126, Theorem 4.5.6, p. 123], [7, Problem 1.5.3, p. 16 and Lemma 3.5.1 and Remark 3.5.1, pp. 164–165])

$$\lim_{t \rightarrow +0} \frac{l_C(t) - l_C(0)}{l_A} = \cos(\angle AP_F C). \quad (2.10)$$

Similarly, let the point D , the length parameterized curve $c(t)$ and the point P_F that belongs in $c(t)$ be given in the neighborhood W_{P_F} . Taking into account (2.4) and (2.5), for $Q = D$, we have

$$\lim_{t \rightarrow +0} \frac{l_D(t) - l_D(0)}{l_A} = \cos(\angle AP_F D). \quad (2.11)$$

Replacing (2.9)–(2.11) in (2.8), we obtain

$$w_A + w_B \cos(\angle AP_F B) = -w_C \cos(\angle AP_F C) - w_D \cos(\angle AP_F D). \quad (2.12)$$

Similarly, working cyclically and differentiating (1.1) with respect to l_B , and choosing the parametrization $l_B(t') = t'$, by differentiating (1.1) with respect to l_C and by choosing the parametrization $l_C(t'') = t''$ and by differentiating (1.1) with respect to l_D and choosing the parametrization $l_D(t''') = t'''$, we derive three relations, respectively,

$$w_A \cos(\angle BP_F A) + w_B = -w_C \cos(\angle BP_F C) - w_D \cos(\angle BP_F D), \quad (2.13)$$

$$w_A \cos(\angle CP_F A) + w_B \cos(\angle CP_F B) = -w_C - w_D \cos(\angle CP_F D), \quad (2.14)$$

$$w_A \cos(\angle DP_F A) + w_B \cos(\angle DP_F B) = -w_C \cos(\angle DP_F C) - w_D. \quad (2.15)$$

The equations (2.12)–(2.15) could be written in a unified form

$$\sum_{Q \in \{A, B, C, D\}} w_Q \cos(\angle QP_F R) = 0 \quad \text{for } R \in \{A, B, C, D\}.$$

The points of M are distinguished into three categories (see [1, p. 386]):

- (1) conical points at which the tangent cone has complete angle less than 2π ,
- (2) edge points at which the tangent cone is dihedral angle,
- (3) "smooth" points at which the tangent cone is a plane.

The point P_F which is the intersection of four prescribed shortest arcs in M can be only a smooth point and it cannot be a conical point or an edge point, due to a result of A.D. Alexandrov in [1, Theorem 5, p. 135]. Therefore, the complete angle of P_F is given by the following formula:

$$\angle AP_FB + \angle BP_FC + \angle CP_FD + \angle DP_FA = 2\pi. \quad (2.16)$$

Taking into account the 4-inverse w. F-T problem, the weights $(w_R)_{ABCD} = w_R$ for $R \in \{A, B, C, D\}$ become variable weights which satisfy (2.12) – (2.15).

Taking into consideration the trigonometric identity

$$1 - \cos(\angle AP_FB) \cos(\angle BP_FA) = -\sin(\angle AP_FB) \sin(\angle BP_FA)$$

and the orientation of the angles with respect to P_F taken counterclockwise (see Fig. 1) we solve the linear system (2.12) – (2.13) with respect to w_A and w_B :

$$w_A \sin(\angle BP_FA) + w_C \sin(\angle BP_FC) + w_D \sin(\angle BP_FD) = 0, \quad (2.17)$$

$$w_B \sin(\angle AP_FB) + w_C \sin(\angle AP_FC) + w_D \sin(\angle AP_FD) = 0. \quad (2.18)$$

Similarly, solving the linear system (2.12) and (2.14) with respect to w_A , w_C , we obtain

$$w_A \sin(\angle CP_FA) + w_B \sin(\angle CP_FB) + w_D \sin(\angle CP_FD) = 0. \quad (2.19)$$

We write (2.19) in the following form:

$$\left(\frac{w_B}{w_A}\right)_{ABCD} = -\frac{\sin(\angle CP_FA)}{\sin(\angle CP_FB)} \left[\left(\frac{w_D}{w_A}\right)_{ABCD} \frac{\sin(\angle CP_FD)}{\sin(\angle CP_FA)} + 1 \right].$$

If we set $(w_D)_{ABCD} = 0$, in (2.17), (2.18) and (2.19), we obtain a solution of the 3-inverse w. F-T problem for the triangle $\triangle ABC$ (see [10], for a solution of this problem on a C^2 -regular surface) and we get

$$-\frac{\sin(\angle CP_FA)}{\sin(\angle CP_FB)} = \left(\frac{w_B}{w_A}\right)_{ABC}.$$

Similarly, if we set $(w_B)_{ABCD} = 0$, in (2.17) – (2.19), we obtain a solution of the 3-inverse w. F-T problem for the triangle $\triangle ACD^*$, where $(D^*$ is the symmetric point of D with respect to P_F , and we get

$$-\left(\frac{w_A}{w_D}\right)_{ACD^*} = \frac{\sin(\angle CP_FD^*)}{\sin(\angle CP_FA)} = -\frac{\sin(\angle CP_FD)}{\sin(\angle CP_FA)} = \left(\frac{w_A}{w_D}\right)_{ACD},$$

which gives (2.1).

We proceed by deriving (2.2). We write (2.17) in the form

$$\left(\frac{w_C}{w_A}\right)_{ABCD} = -\frac{\sin(\angle BP_FA)}{\sin(\angle BP_FC)} \left(1 + \left(\frac{w_D}{w_A}\right)_{ABCD} \frac{\sin(\angle BP_FD)}{\sin(\angle BP_FA)} \right).$$

If we set $(w_D)_{ABCD} = 0$, in (2.17) – (2.19), we obtain a solution of the 3-inverse w. F-T problem for the triangle $\triangle ABC$ and we get

$$\left(\frac{w_C}{w_A}\right)_{ABC} = -\frac{\sin(\angle BP_FA)}{\sin(\angle BP_FC)}.$$

Similarly, if we set $(w_C)_{ABCD} = 0$, in (2.17) – (2.19), we obtain a solution of the 3-inverse w. F-T problem for the triangle $\triangle ABD$, and we get

$$\left(\frac{w_A}{w_D}\right)_{ABD} = -\frac{\sin(\angle BP_FD)}{\sin(\angle BP_FA)},$$

which yields (2.2). \square

As a direct consequence of Theorem 1 we obtain

Corollary 1. *If $\sum_{ABCD} w = \sum_{ABC} w = \sum_{ABD} w = \sum_{ACD} w = \sum_{BCD} w$, where*

$$\sum_{ABCD} w := (w_A)_{ABCD} \left(1 + \frac{w_B}{w_A} + \frac{w_C}{w_A} + \frac{w_D}{w_A}\right)_{ABCD},$$

then

$$(w_i)_{ABCD} = a_i(w_D)_{ABCD} + b_i, \quad i \in \{A, B, C\}, \quad (2.20)$$

where

$$\begin{aligned} (a_A, b_A) &= \left(\frac{\left(\frac{w_A}{w_D}\right)_{ACD} \left(\frac{w_B}{w_A}\right)_{ABC} + \left(\frac{w_A}{w_D}\right)_{ABD} \left(\frac{w_C}{w_A}\right)_{ABC} - 1}{1 + \left(\frac{w_B}{w_A}\right)_{ABC} + \left(\frac{w_C}{w_A}\right)_{ABC}}, (w_A)_{ABC} \right), \\ (a_B, b_B) &= \left(a_A \left(\frac{w_B}{w_A}\right)_{ABC} - \left(\frac{w_A}{w_D}\right)_{ACD} \left(\frac{w_B}{w_A}\right)_{ABC}, (w_B)_{ABC} \right), \\ (a_C, b_C) &= \left(a_A \left(\frac{w_C}{w_A}\right)_{ABC} - \left(\frac{w_A}{w_D}\right)_{ABD} \left(\frac{w_C}{w_A}\right)_{ABC}, (w_C)_{ABC} \right). \end{aligned} \quad (2.21)$$

We continue by proving the main result, which we call a *plasticity principle of convex quadrilaterals on M* :

Theorem 2. *Given four shortest arcs which meet at the weighted Fermat-Torricelli point P_F and their endpoints form a convex quadrilateral on M and the weighted Fermat-Torricelli point belongs to the interior of this convex quadrilateral, an increase of the weight that corresponds to a shortest arc causes a decrease to the two weights that correspond to the two neighboring shortest arcs and an increase to the weight that corresponds to the opposite shortest arc.*

Proof of Theorem 2: We take into account the three plasticity equations of Theorem 1 and Corollary 1 which are derived by applying the conditions $\sum_{ABCD} w := (w_A)_{ABCD} \left(1 + \frac{w_B}{w_A} + \frac{w_C}{w_A} + \frac{w_D}{w_A}\right)_{ABCD}$ and $\sum_{ABCD} w = \sum_{ABC} w = \sum_{ABD} w = \sum_{ACD} w = \sum_{BCD} w = \text{const}$ such that (2.20)–(2.21) hold. We prove that $a_A, a_C < 0$ and $a_B > 0$. We calculate the coefficient a_A :

$$\begin{aligned} a_A &= \frac{\left(\frac{w_A}{w_D}\right)_{ACD} \left(\frac{w_B}{w_A}\right)_{ABC} + \left(\frac{w_A}{w_D}\right)_{ABD} \left(\frac{w_C}{w_A}\right)_{ABC} - 1}{1 + \left(\frac{w_B}{w_A}\right)_{ABC} + \left(\frac{w_C}{w_A}\right)_{ABC}} \\ &= \frac{\frac{\sin(\angle CP_F D)}{\sin(\angle CP_F A)} \frac{\sin(\angle CP_F A)}{\sin(\angle CP_F B)} + \frac{\sin(\angle BP_F D)}{\sin(\angle BP_F A)} \frac{\sin(\angle BP_F A)}{\sin(\angle BP_F C)} - 1}{\frac{\text{const}}{(w_A)_{ABC}}} \\ &= \frac{\frac{\sin(\angle C^* P_F D)}{\sin(\angle C^* P_F B)} - \frac{\sin(\angle BP_F D)}{\sin(\angle BP_F C^*)} - 1}{\frac{\text{const}}{(w_A)_{ABC}}} = \frac{-\left(\frac{w_B}{w_D}\right)_{BC^*D} + \left(\frac{w_{C^*}}{w_D}\right)_{BC^*D} - 1}{\frac{\text{const}}{(w_A)_{ABC}}} \\ &= \frac{-\left(\frac{w_B}{w_D}\right)_{BCD} - \left(\frac{w_C}{w_D}\right)_{BCD} - 1}{\frac{\text{const}}{(w_A)_{ABC}}} = -\frac{(w_A)_{ABC}}{(w_D)_{BCD}} < 0, \end{aligned}$$

because $(w_A)_{ABC}$ and $(w_D)_{BCD}$ are positive numbers. The point C^* is the symmetric point of C with respect to P_F such that $l_{P_F}(C) = l_{P_F}(C^*)$. Taking into account that P_F is located at the interior of the triangle $\triangle BC^*D$, we have

$$(w_B)_{BC^*D} + (w_{C^*})_{BC^*D} + (w_D)_{BC^*D} = c,$$

where $(w_R)_{BC^*D}$ are positive numbers for $R \in \{B, C^*, D\}$.

Concerning the triangle $\triangle BCD$, P_F is not located at the interior of P_F and by the relation

$$(w_B)_{BCD} + (w_C)_{BCD} + (w_D)_{BCD} = c,$$

we obtain

$$(w_C)_{BCD} = -(w_{C^*})_{BC^*D} < 0, \quad (w_B)_{BCD}, (w_D)_{BCD} > 0.$$

Therefore,

$$a_C = a_A \left(\frac{w_C}{w_A} \right)_{ABC} - \left(\frac{w_A}{w_D} \right)_{ABD} \left(\frac{w_C}{w_A} \right)_{ABC} < 0.$$

because P_F is located at the interior of the triangles $\triangle ABC$, $\triangle ABD$, which makes the weights $(w_A)_{ABC}$, $(w_C)_{ABC}$, $(w_A)_{ABD}$, $(w_D)_{ABD}$ positive numbers.

We shall show $a_B > 0$. From the 4-inverse w. F-T condition of Problem 2 we have

$$(w_A)_{ABCD} = c - (w_B)_{ABCD} - (w_C)_{ABCD} - (w_D)_{ABCD}. \quad (2.22)$$

Replacing (2.22) in (2.19), we get

$$(w_B)_{ABCD} (\sin(\angle CP_F B) - \sin(\angle CP_F A)) + 2(w_C)_{ABCD} \sin(\angle AP_F C) = w_D (\sin(\angle CP_F A) - \sin(\angle CP_F D)) - c \sin(\angle CP_F A). \quad (2.23)$$

From (2.18) and (2.23) we derive that

$$(w_B)_{ABCD} = \frac{\sin(\angle CP_F A) - \sin(\angle CP_F D) + 2 \sin(\angle AP_F D)}{\sin(\angle CP_F B) - \sin(\angle CP_F A) - 2 \sin(\angle AP_F B)} (w_D)_{ABCD} - \frac{c \sin(\angle CP_F A)}{\sin(\angle CP_F B) - \sin(\angle CP_F A) - 2 \sin(\angle AP_F B)}. \quad (2.24)$$

Taking into account the counterclockwise orientation of angles with respect to P_F (see Fig. 1), we get

$$\sin(\angle CP_F B) - \sin(\angle CP_F A) - 2 \sin(\angle AP_F B) < 0,$$

because

$$\sin(\angle CP_F B) < 0, \quad -\sin(\angle CP_F A) < 0, \quad -\sin(\angle AP_F B) < 0.$$

Similarly, taking into account the counterclockwise orientation of angles with respect to P_F (see Fig. 1), we get

$$\begin{aligned} & \sin(\angle CP_F A) - \sin(\angle CP_F D) + 2 \sin(\angle AP_F D) \\ &= -\sin(\angle AP_F C) - \sin(\angle CP_F D) + \sin(\angle AP_F C + \angle CP_F D) + \sin(\angle AP_F D) \\ &= \sin(\angle AP_F C)(\cos(\angle CP_F D) - 1) + \sin(\angle CP_F D)(\cos(\angle AP_F C) - 1) + \sin(\angle AP_F D) \\ &\leq \sin(\angle AP_F D) < 0. \end{aligned}$$

Therefore, we have

$$a_B = \frac{\sin(\angle CP_F A) - \sin(\angle CP_F D) + 2 \sin(\angle AP_F D)}{\sin(\angle CP_F B) - \sin(\angle CP_F A) - 2 \sin(\angle AP_F B)} > 0$$

that completes the proof. \square

A numerical verification of the plasticity principle is given in [9, Examples 4.7, 4.10, pp. 418–419].

Remark 1. Concerning the plasticity of convex quadrilaterals on M , we have assumed that P_F is located at the interior of $\triangle ABC$. Then a fourth shortest arc "grows" from P_F such that its length reaches $l_{P_F}(D)$ and $ABCD$ is a convex quadrilateral on M . Therefore, we consider that the 4-inverse w.F-T problem is derived by the 3-inverse w.F-T problem on M .

We mention two particular cases (Proposition 1, Example 1) such that P_F belongs to at least one of the diagonals (shortest arc) of $ABCD$ where the 4-inverse w.F-T problem is not derived by the 3-inverse w.F-T problem like in Theorem 2.

Proposition 1. *If P_F belongs to the shortest arc (B, D) and does not necessarily belong to the shortest arc (A, C) , then the plasticity principle of the convex quadrilateral $ABCD$ on M holds.*

Proof of Proposition 1: Assuming that $P_F \in (B, D) \setminus ([B, D] \cap [A, C])$, the angle $\angle BP_F D$ is π and from (2.17), we have

$$w_A = -\frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)} w_C. \quad (2.25)$$

Replacing (2.25) to the inverse condition

$$w_A + w_B + w_C + w_D = c,$$

we get

$$\left(1 - \frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)}\right) w_C + w_B = c - w_D. \quad (2.26)$$

Replacing (2.25) in (2.19), we have

$$-\frac{\sin(\angle CP_F A) \sin(\angle BP_F C)}{\sin(\angle BP_F A)} w_C + w_B \sin(\angle CP_F B) = -\sin(\angle CP_F D) w_D. \quad (2.27)$$

Solving (2.26) and (2.27) with respect to w_C and w_A , we derive that

$$w_C = x_C w_D + y_C, \quad w_B = x_B w_D + y_B,$$

where x_C, y_C, x_B, y_B are constant numbers such that:

$$x_C = \frac{-\sin(\angle CP_F B) + \sin(\angle CP_F D)}{\text{Det}}, \quad (2.28)$$

$$x_B = \frac{-\sin(\angle CP_F D) \left(1 - \frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)}\right) + \sin(\angle CP_F A) \frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)}}{\text{Det}}, \quad (2.29)$$

and Det is the determinant of (2.26) and (2.27):

$$\text{Det} = \left(1 - \frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)}\right) \sin(\angle CP_F B) + \sin(\angle CP_F A) \frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)}.$$

We will prove that $x_C < 0$ and $x_B > 0$.

The determinant Det of (2.26) and (2.27) is a negative because $\sin(\angle CP_F B) < 0$, $\frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)} < 1$, $\sin(\angle BP_F A) < 0$, $\sin(\angle CP_F A) > 0$, $\sin(\angle BP_F C) > 0$. Taking into account that the numerator of (2.28)

$$-\sin(\angle CP_F B) + \sin(\angle CP_F D) > 0$$

because

$$-\sin(\angle CP_F B) > 0, \quad \sin(\angle CP_F D) > 0,$$

we obtain that $x_C < 0$. Taking into account that $\text{Det} < 0$ and that the numerator of (2.29)

$$-\sin(\angle CP_F D) \left(1 - \frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)}\right) + \sin(\angle CP_F A) \frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)} < 0$$

because

$$-\sin(\angle CP_F D) \left(1 - \frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)}\right) < 0,$$

and that

$$\sin(\angle CP_F A) \frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)} < 0$$

because $\sin(\angle BP_F A) < 0$ (If P' is the point where the diagonals meet we take $P_F \in [B, D]$ to be located closer to B with respect to P'), we obtain that $x_B > 0$.

The following equations

$$w_C = x_C w_D + y_C, \quad w_B = x_B w_D + y_B,$$

where $x_C < 0$ and $x_B > 0$, give the plasticity principle of $ABCD$, because by increasing w_D , the weight w_C will be decreased and the weight w_B will be increased and considering the relation (2.25) w_A will also be decreased because $-\frac{\sin(\angle BP_F C)}{\sin(\angle BP_F A)}$ is a positive number. \square

Example 1. If P_F is the intersection of the two diagonals (B, D) and (A, C) , then the plasticity principle of $ABCD$ on M holds.

To show this directly, assume that $P_F \in [B, D] \cap [A, C]$, and get

$$\angle BP_F D = \angle CP_F A = \pi.$$

From (2.17), (2.19) we derive $w_A = w_C$ and $w_B = w_D$, respectively.

Replacing these two relations in the 4-inverse condition of Problem 2, we have

$$2w_A + 2w_D = c,$$

where c is a positive real number. By the last relation, an increase in $w_D = w_B$ will cause a decrease in $w_A = w_C$, otherwise the inverse condition will not hold.

Remark 2. We would like to mention a reformulation of the 4 inverse w.F-T problem for a convex quadrilateral $ABCD$ on \mathbb{R}^2 , which is communicated to the author by Professor Dr. Vladimir Rovenski.

Let $\Pi = \{w_A + w_B + w_C + w_D = 1\}$ (say, $c = 1$) be a 3-plane in the linear space \mathbb{R}^4 of variables (w_A, w_B, w_C, w_D) . Denote $v_1 = PA/|PA|$, $v_2 = PB/|PB|$, $v_3 = PC/|PC|$, $v_4 = PD/|PD|$ – the unit vectors in \mathbb{R}^2 . The linear system $w_A v_1 + w_B v_2 + w_C v_3 + w_D v_4 = 0$ (of two equations with respect to (x, y) given coordinates) determines a straight line $w_R(t) = w_R(0) + t\vec{e}$ in Π , where $e = (e_1, e_2, e_3, e_4)$ is a direction vector of the line.

Taking the inner product of $w_A v_1 + w_B v_2 + w_C v_3 + w_D v_4 = 0$ with respect to the vector v_i for $i = 1, 2, 3, 4$, we obtain (2.12)–(2.15).

Taking the exterior product of $w_A v_1 + w_B v_2 + w_C v_3 + w_D v_4 = 0$ with respect to the vector v_i for $i = 1, 2, 3, 4$, we obtain (2.17)–(2.19).

3. THE PLASTICITY OF QUADRILATERALS ON A MANIFOLD OF POSITIVE CURVATURE

We give some evolutionary structures for quadrilaterals on a manifold with positive curvature. A metric of positive curvature is intrinsic and the sum of the lower angles of every sufficiently small convex triangle cannot be less than π (see [1, p. 279]). The evolutionary structure for quadrilaterals is derived by the plasticity equations and the 4-inverse Fermat-Torricelli problem for the following two cases:

(1) A manifold M' which is obtained by gluing two triangles that exist on a two-dimensional sphere with constant Gaussian curvature K_1 and two triangles that exist on a two-dimensional sphere of constant Gaussian curvature K_2 , for $K_1 < K_2$, and the sum of angles meeting at the Fermat-Torricelli point P_F of the quadrilateral equals 2π ,

(2) A manifold M'' which is obtained by gluing two triangles that exist on a complete convex surface of bounded specific curvature M ($K_1 < \kappa < K_2$) and two triangles that exist on a two-dimensional sphere of constant Gaussian curvature K_1 , and K_2 , respectively, and the sum of angles meeting at the Fermat-Torricelli point P_F of the quadrilateral equals 2π .

These two cases form a manifold with metric of positive curvature because they are specific cases of the gluing theorem of A.D. Alexandrov (see [1, pp. 278–279]) which states that:

If a manifold is obtained by gluing a finite number of polygons with metric of positive curvature so that, at each vertex, the sum of angles of these polygons meeting at it is no greater than 2π , then the metric on the whole manifold is also a metric of positive curvature.

A further generalization of the gluing theorem of A.D. Alexandrov was made by Yu. Reshetnyak concerning the gluing of Cartan-Alexandrov-Toponogov spaces with curvature (in the sense of A.D. Alexandrov) bounded from above by a real number K_2 ($\text{CAT}(K_2)$) spaces along proper convex subsets is a $\text{CAT}(K_2)$ space (see [7, pp. 188–189 and footnote 15] and [2]).

We mention the definition of a K_0 -plane and the comparison triangle of a triangle on a two-dimensional surface.

If K_0 denotes the constant Gaussian curvature of a surface M , then M is called the K_0 -plane.

If $K_0 < 0$, the K_0 -plane is a Lobachevski (hyperbolic) plane H^2 .

If $K_0 = 0$, the K_0 -plane is an Euclidean plane \mathbb{R}^2 .

If $K_0 > 0$, the K_0 -plane is an open hemisphere S^2 of radius $\frac{1}{\sqrt{K_0}}$.

A *comparison triangle* on a K_0 -plane denoted by $(\triangle ABC)_{K_0}$ of $\triangle ABC$ on a two-dimensional surface is a triangle whose corresponding sides have equal lengths: $l_A(B) = l_A(B)$, $l_B(C) = l_B(C)$, $l_A(C) = l_A(C)$ ([7, pp. 185, 188]). The existence of the comparison triangle $(\triangle ABC)_{K_0}$, for $K_0 > 0$, is given by the condition that the perimeter of the triangle is not greater than $\frac{2\pi}{\sqrt{K_0}}$.

The following corollary is given on M' and M'' which are particular cases of M considered in Theorem 1.

Corollary 2. Consider the 4-inverse w - F - T problem on M^q for $q \in \{\iota, \iota'\}$. The following equations point out the plasticity of quadrilaterals $A^q B^q C^q D^q$ and on M^q :

$$\left(\frac{w_{B^q}}{w_{A^q}}\right)_{A^q B^q C^q D^q} = \left(\frac{w_{B^q}}{w_{A^q}}\right)_{A^q B^q C^q} \left[1 - \left(\frac{w_{D^q}}{w_{A^q}}\right)_{A^q B^q C^q D^q} \left(\frac{w_{A^q}}{w_{D^q}}\right)_{A^q C^q D^q}\right], \quad (3.1)$$

$$\left(\frac{w_{C^q}}{w_{A^q}}\right)_{A^q B^q C^q D^q} = \left(\frac{w_{C^q}}{w_{A^q}}\right)_{A^q B^q C^q} \left[1 - \left(\frac{w_{D^q}}{w_{A^q}}\right)_{A^q B^q C^q D^q} \left(\frac{w_{A^q}}{w_{D^q}}\right)_{A^q B^q D^q}\right], \quad (3.2)$$

and

$$(w_{A^q})_{A^q B^q C^q D^q} + (w_{B^q})_{A^q B^q C^q D^q} + (w_{C^q})_{A^q B^q C^q D^q} + (w_{D^q})_{A^q B^q C^q D^q} = \text{const} \quad (3.3)$$

where the weight $(w_R)_{A^q B^q C^q D^q}$ corresponds to the vertex R that lie on the shortest arc $P_F R$, $R \in \{A^q, B^q, C^q, D^q\}$ and the weight $(w_S)_{SLN}$ corresponds to the vertex S that lie on the shortest arc $P_F S$ regarding the triangle $\triangle SLN$, for $S, L, N \in \{A^q, B^q, C^q, D^q\}$ and $S \neq L \neq N$, for $q \in \{\iota, \iota'\}$.

Proof of Corollary 2 (Case M'): Gluing the comparison triangles $(\triangle A' P_F D')_{K_1}$, $(\triangle D' P_F C')_{K_2}$, $(\triangle C' P_F B')_{K_3}$, $(\triangle B' P_F A')_{K_4}$ of $\triangle A P_F D$, $\triangle D P_F C$, $\triangle C P_F B$, and $\triangle B P_F A$, respectively, we obtain the following angular relations on M' :

$$\begin{aligned} \angle A' P_F D' &= \angle A P_F D - \epsilon_1, & \angle D' P_F C' &= \angle D P_F C - \epsilon_2, \\ \angle C' P_F B' &= \angle C P_F B + \epsilon_3, & \angle B' P_F A' &= \angle B P_F A + \epsilon_4, \end{aligned}$$

(where ϵ_i , are non-negative real numbers for $i = 1, 2, 3, 4$) such that

$$\angle A' P_F D' + \angle D' P_F C' + \angle C' P_F B' + \angle B' P_F A' = 2\pi.$$

The angular relations hold due to a result of A.D. Alexandrov also known as the angle comparison theorem which states that:

Let T be a triangle on a complete convex surface of bounded specific curvature and let triangle T_1, T_2 be triangles on the two-dimensional spheres S_{K_1} and S_{K_2} with sides of the same length as T . If $\alpha, \alpha_1, \alpha_2$ are corresponding angles of these triangles then $\alpha_1 \leq \alpha \leq \alpha_2$ (see [1, Theorem 4, p. 377, p. 54, case 1]).

We apply the following general result which was given in [4, Corollary 4.5.7, p. 125, Remark 4.5.8, p. 126 and Theorem 4.5.6, p. 123] that deals with a rule for differentiating the length of a continuous family of shortest arcs connecting Q to points $c(t)$, for $Q \in \{A, B, C\}$ and the existence of the right derivative $\frac{dl_Q}{dt}$, on a non-negatively curved complete locally compact space X (M' a subset of X):

Let $c(t) : [0, T] \rightarrow M$ be a geodesic parameterized by arclength, Q a fixed point in M , $Q \neq c(0)$. Then the function $t \rightarrow l_Q(t) = |Qc(t)|$ has the right derivative and

$$\lim_{t \rightarrow +0} \frac{l_Q(t) - l_Q(0)}{t} = \cos(\pi - \varphi'_{c, \min}(t)),$$

where $\varphi'_{c, \min}(t)$ is the infimum (minimum) of angles between $c(t)$ and the shortest arcs connecting $c(0)$ to Q . Taking into considerations the parameterization $l_Q(t) = t$, for $Q \in \{A', B', C', D'\}$ and differentiating the objective function (1.1) with respect to t and following the same process in the proof of Theorem 1, we get

$$\sum_{Q \in \{A', B', C', D'\}} w_Q \cos(\angle Q P_F R) = 0 \quad \text{for } Q, R \in \{A', B', C', D'\} \text{ and } W_A \subset M.$$

Applying the same steps that have been used in the proof of Theorem 1, we obtain (3.1) and (3.2). This system of equations induces a *comparative plasticity* between the plasticity of $ABCD$ on M and $A'B'C'D'$ on M' because we may compare the

plasticity equations on a convex surface M with the plasticity equations on a convex surface M' such that the given directions with respect to P_F vary in a specific way that depend on $\angle R'P_FS'$ for $R', S' \in \{A', B', C', D'\}$:

$$\begin{aligned}\angle A'P_FD' &= \angle AP_FD - \epsilon_1, & \angle D'P_FC' &= \angle DP_FC - \epsilon_2, \\ \angle C'P_FB' &= \angle CP_FB + \epsilon_3, & \angle B'P_FA' &= \angle BP_FA + \epsilon_4,\end{aligned}$$

and every ratio $\frac{(\frac{w_{R'}}{w_{S'}})_{A'B'C'D'}}{(\frac{w_R}{w_S})_{ABCD}}$ depend on the values of ϵ_i for $i = 1, 2, 3, 4$ and $R, S \in \{A, B, C, D\}$.

For instance by taking into account that

$$a_A + a_B + a_C = -1,$$

and let a_A, a_C be increased by $\delta a_A, \delta a_C > 0$ then a_B will be decreased by $\delta a_B < 0$ such that

$$a_A + \delta a_A + a_B + \delta a_B + a_C + \delta a_C = -1,$$

or

$$\delta a_A + \delta a_B + \delta a_C = 0,$$

where $a_{R'} = a_R + \delta a_R$, where a_R is taken by Corollary 1, for $R \in \{A, B, C\}$. \square

Proof of Corollary 2 (Case M''): Taking into consideration the gluing of the triangles $\triangle AP_FD, \triangle DP_FC$ which exist on M with the two comparison triangles $(\triangle C'P_FB')_{K_1}, (\triangle B'P_FA')_{K_2}$ of $\triangle CP_FB, \triangle BP_FA$, respectively, we obtain the following angular relations on M'' :

$$\begin{aligned}\angle A''P_FD'' &= \angle AP_FD, & \angle D''P_FC'' &= \angle DP_FC, \\ \angle C''P_FB'' &= \angle CP_FB - \epsilon_3, & \angle B''P_FA'' &= \angle BP_FA + \epsilon_4,\end{aligned}$$

(where ϵ_i are non-negative real numbers for $i = 3, 4$) such that

$$\angle A''P_FD'' + \angle D''P_FC'' + \angle C''P_FB'' + \angle B''P_FA'' = 2\pi.$$

The angular relations hold due to a result of A.D. Alexandrov also known as the angle comparison theorem (see [1, Theorem 4, p. 377, p. 54, case 1]).

Applying the same process that have been used in the proof of Theorem 1, we derive the desired plasticity equations (3.1) and (3.2). \square

Remark 3. The comparative plasticity between the surfaces M and M' or M'' given in Corollary 2 approaches some type of formulation of the plasticity of a mathematical "gibbosity" (part of a sphere on a convex surface) and it might be considered as a useful tool for future medical applications in the area of robotics.

4. A SYMMETRIZATION OF WEIGHTED QUADRILATERALS ON A SURFACE OF GAUSSIAN CURVATURE BOUNDED ABOVE BY A POSITIVE NUMBER

We introduce a new symmetrization technique which transforms some classes of weighted convex quadrilaterals on a C^2 complete convex surface of bounded Gaussian curvature M to parallelograms which lie on the same tangent plane that is defined at the weighted Fermat-Torricelli point of the corresponding quadrilateral.

We give two classes of parallelograms which characterize the evolution of convex quadrilaterals which means that the weights satisfy the plasticity equations of Theorem 1

The variable *Gaussian curvature* K is positive and bounded below by a real positive number $K_1 : K_1 < K$, then the geodesic distance of length not greater than $\frac{\pi}{\sqrt{K_1}}$ and is unique, and an estimate of the *injectivity radius* $r_i = \inf(r_i(P) : P \in M)$ of M is given by the inequality $r_i \leq \frac{\pi}{\sqrt{K_1}}$, see [6], [7, Theorem 3.5.2].

We consider a convex quadrilateral $A^\circ B^\circ C^\circ D^\circ$ that belongs on a neighborhood $W_A \subset M$, where W_A is a subset of a geodesic circle with center A and radius r_i , and the perimeter of the quadrilateral is smaller than $2r_i$.

Suppose that we select w_R for $R \in \{A^\circ B^\circ C^\circ D^\circ\}$ (Fig. 2) such that P_F is located at the interior domain of $A^\circ B^\circ C^\circ D^\circ$ and some inequalities are also satisfied

$$w_{B^\circ} > w_{A^\circ} > w_{D^\circ} > w_{C^\circ}.$$

We consider a quadrilateral $A^\circ B^\circ C^\circ D^\circ$ that belongs on a neighborhood $W_A \subset M$, where W_A is a subset of a geodesic circle with center A and radius r_i , and the perimeter of the quadrilateral is smaller than $2r_i$.

Suppose that we select w_R for $R \in \{A^\circ B^\circ C^\circ D^\circ\}$ (Fig. 2) such that P_F is located at the interior domain of $A^\circ B^\circ C^\circ D^\circ$.

Proposition 2. *A symmetrization of $A^\circ B^\circ C^\circ D^\circ$ with respect to P_F is the parallelogram $A'^\star B'C'^\star D'$, where A'^\star, C'^\star are the symmetric points of A', C' with respect to P_F (see Figs. 2 and 3),*

$$|P_F R| = |\exp_{P_F}^{-1}(R)| = |P_F R'| = w_R$$

where R lies on the geodesic $P_F R^\circ$ for $R \in \{A, B, C, D\}$ and $R' \in \{A', B', C', D'\}$.

Proof of Proposition 2: The geodesic arcs l_R for $R \in \{A^\circ, B^\circ, C^\circ, D^\circ\}$ belong to W_A and they are shortest arcs and any shortest arc is a geodesic (see [7, Lemma 3.5.2, p. 165 and Theorem 3.5.2, p. 167]). A particular case of the differentiation of the length of a continuous family of shortest arcs given in [4, Corollary 4.5.7, p. 125] is the differentiation of the length of a C^2 family of geodesic arcs given in [7, Lemma 3.5.1, p. 164 and Remark 3.5.1].

Let the geodesic arcs $l_{B^\circ}, l_{C^\circ}, l_{D^\circ}$ be expressed as functions of l_{A° .

$$l_{B^\circ} = l_{B^\circ}(l_{A^\circ}), \quad l_{C^\circ} = l_{C^\circ}(l_{A^\circ}), \quad l_{D^\circ} = l_{D^\circ}(l_{A^\circ}). \quad (4.1)$$

From (4.1) and (1.1) the following equation is obtained replacing $R \rightarrow R^\circ$, for $R^\circ \in \{A^\circ, B^\circ, C^\circ, D^\circ\}$

$$w_{A^\circ} l_{A^\circ} + w_{B^\circ} l_{B^\circ}(l_{A^\circ}) + w_{C^\circ} l_{C^\circ}(l_{A^\circ}) = \min \quad (4.2)$$

Let $c(t) : [0, T] \rightarrow M$ be a geodesic parameterized by arclength, Q a fixed point on M , $Q \neq c(0)$. Then the function $t \rightarrow l_Q(t) = |Qc(t)|$ has the right derivative and

$$\lim_{t \rightarrow +0} \frac{l_Q(t) - l_Q(0)}{t} = \cos(\pi - \varphi'_c(t)), \quad (4.3)$$

where $\varphi'_c(t)$ is the angle between $c(t)$ and the geodesic arc connecting $c(0)$ to Q .

We choose the parametrization

$$l_{A^\circ}(t) = t, \quad (4.4)$$

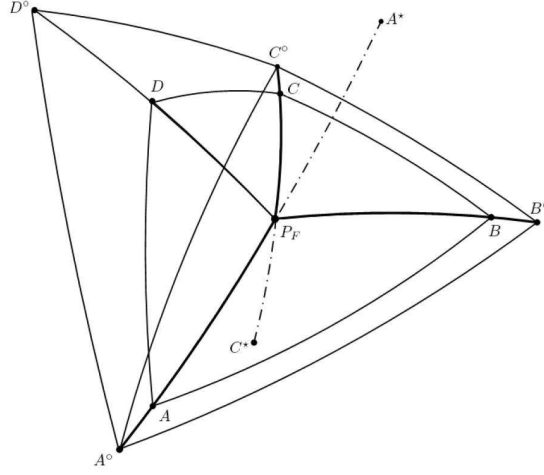


FIGURE 2.

Taking into account that P_F is the w. F-T point of $A^\circ B^\circ C^\circ D^\circ$, we derive the (right) derivative of (4.2) with respect to the variable l_{A° and considering (4.4), we get

$$\begin{aligned} w_{A^\circ} + w_{B^\circ} \lim_{t \rightarrow +0} \frac{l_{B^\circ}(t) - l_{B^\circ}(0)}{l_{A^\circ}(t)} + w_{C^\circ} \lim_{t \rightarrow +0} \frac{l_{C^\circ}(t) - l_{C^\circ}(0)}{l_{A^\circ}(t)} \\ + w_{D^\circ} \lim_{t \rightarrow +0} \frac{l_{D^\circ}(t) - l_{D^\circ}(0)}{l_{A^\circ}(t)} = 0. \end{aligned} \quad (4.5)$$

From (4.3) and (4.4), we get

$$\lim_{t \rightarrow +0} \frac{l_{B^\circ}(t) - l_{B^\circ}(0)}{l_{A^\circ}(t)} = \cos(\angle A^\circ P_F B^\circ). \quad (4.6)$$

Let the point C° , the length parameterized curve $c(t)$ and the point P_F that belongs in $c(t)$ be given in the neighborhood W_{P_F} . Taking into account (4.3) and (4.4), we have ([7, Lemma 3.5.1, p. 164 and Remark 3.5.1, p. 165])

$$\lim_{t \rightarrow +0} \frac{l_{C^\circ}(t) - l_{C^\circ}(0)}{l_{A^\circ}} = \cos(\angle A^\circ P_F C^\circ). \quad (4.7)$$

Similarly, let the point D° , the length parameterized curve $c(t)$ and the point P_F that belongs in $c(t)$ be given in the neighborhood W_A . Taking into account (4.3) and (4.4), we obtain

$$\lim_{t \rightarrow +0} \frac{l_{D^\circ}(t) - l_{D^\circ}(0)}{l_{A^\circ}} = \cos(\angle A^\circ P_F D^\circ). \quad (4.8)$$

Replacing (4.6) – (4.8) in (4.5), we obtain

$$w_{A^\circ} + w_{B^\circ} \cos(\angle A^\circ P_F B^\circ) = -w_{C^\circ} \cos(\angle A^\circ P_F C^\circ) - w_{D^\circ} \cos(\angle A^\circ P_F D^\circ). \quad (4.9)$$

Similarly, working cyclically and differentiating (1.1) with respect to l_{B° , and choosing the parametrization $l_{B^\circ}(t') = t'$, differentiating (1.1) with respect to l_{C° and choosing the parametrization $l_{C^\circ}(t'') = t''$ and differentiating (1.1) with respect to l_{D° and choosing the parametrization $l_{D^\circ}(t''') = t'''$, we derive three relations,

respectively,

$$w_{A^\circ} \cos(\angle B^\circ P_F A^\circ) + w_{B^\circ} = -w_{C^\circ} \cos(\angle B^\circ P_F C^\circ) - w_{D^\circ} \cos(\angle B^\circ P_F D^\circ), \quad (4.10)$$

$$w_{A^\circ} \cos(\angle C^\circ P_F A^\circ) + w_{B^\circ} \cos(\angle C^\circ P_F B^\circ) = -w_{C^\circ} - w_{D^\circ} \cos(\angle C^\circ P_F D^\circ), \quad (4.11)$$

$$w_{A^\circ} \cos(\angle D^\circ P_F A^\circ) + w_{B^\circ} \cos(\angle D^\circ P_F B^\circ) = -w_{C^\circ} \cos(\angle D^\circ P_F C^\circ) - w_{D^\circ}. \quad (4.12)$$

The equations (4.9)–(4.12) could be written in a unified form

$$\sum_{Q \in \{A^\circ, B^\circ, C^\circ, D^\circ\}} w_Q \cos(\angle Q P_F R) = 0 \quad \text{for } R \in \{A^\circ, B^\circ, C^\circ, D^\circ\}.$$

Consider the 4-inverse w. F-T problem under the condition

$$\sum_{R \in \{A^\circ, B^\circ, C^\circ, D^\circ\}} (w_R)_{A^\circ B^\circ C^\circ D^\circ} = c.$$

Solving the linear system (4.9)–(4.10) with respect to w_{A° and w_{B° , we obtain

$$w_{A^\circ} \sin(\angle B^\circ P_F A^\circ) + w_{C^\circ} \sin(\angle B^\circ P_F C^\circ) + w_{D^\circ} \sin(\angle B^\circ P_F D^\circ) = 0, \quad (4.13)$$

$$w_{B^\circ} \sin(\angle A^\circ P_F B^\circ) + w_{C^\circ} \sin(\angle A^\circ P_F C^\circ) + w_{D^\circ} \sin(\angle A^\circ P_F D^\circ) = 0. \quad (4.14)$$

Similarly, solving the system (4.9) and (4.11) with respect to w_{A° , w_{C° , we obtain

$$w_{A^\circ} \sin(\angle C^\circ P_F A^\circ) + w_{B^\circ} \sin(\angle C^\circ P_F B^\circ) + w_{D^\circ} \sin(\angle C^\circ P_F D^\circ) = 0. \quad (4.15)$$

We write (4.14) in the following form:

$$w_{B^\circ} \sin(\angle A^\circ P_F B^\circ) = -w_{C^\circ} \sin(\angle A^\circ P_F C^\circ) - w_{D^\circ} \sin(\angle A^\circ P_F D^\circ). \quad (4.16)$$

From the derived equations (4.9) – (4.15) we obtain a balancing condition of four tangent vectors at P_F which are located at the tangent plane $T_{P_F}(M)$ at P_F having their weighted sum zero. By this approach, we deduce the invariance property of the w. F-T point P_F for a given convex quadrilateral $A^\circ B^\circ C^\circ D^\circ$ which states that:

Suppose that there is a convex quadrilateral $A^\circ B^\circ C^\circ D^\circ$ on M and a non-negative weight w_R corresponds at each vertex R , for $R \in \{A^\circ B^\circ C^\circ D^\circ\}$. Assume that the w. F-T point P_F point is an interior point of $A^\circ B^\circ C^\circ D^\circ$. If P_F is connected with every vertex R and a point S is selected with a non-negative weight w_R such that S belongs to the geodesic that is defined by the geodesic arc $P_F R$, for $R \in \{A^\circ, B^\circ, C^\circ, D^\circ\}$ and $S \in \{A, B, C, D\}$ and the convex quadrilateral $ABCD$ is constructed such that the corresponding weighted Fermat-Torricelli point P'_F is not a vertex of $ABCD$, then the w. F-T point P'_F is identical with P_F . Squaring both parts of (4.9) and (4.16) and adding the two derived equations, we get

$$w_{A^\circ}^2 + w_{B^\circ}^2 + 2w_{A^\circ} w_{B^\circ} \cos(\angle A^\circ P_F B^\circ) = w_{C^\circ}^2 + w_{D^\circ}^2 + 2w_{C^\circ} w_{D^\circ} \cos(\angle C^\circ P_F D^\circ). \quad (4.17)$$

Applying the same process and exchanging the indices $D^\circ \leftrightarrow B^\circ$ in (4.17), we get

$$w_{A^\circ}^2 + w_{D^\circ}^2 + 2w_{A^\circ} w_{D^\circ} \cos(\angle A^\circ P_F D^\circ) = w_{B^\circ}^2 + w_{C^\circ}^2 + 2w_{B^\circ} w_{C^\circ} \cos(\angle B^\circ P_F C^\circ). \quad (4.18)$$

The invariance property of the w. F-T P_F gives us the ability to transform the initial quadrilateral $A^\circ B^\circ C^\circ D^\circ$ to $ABCD$ and applying the inverse of the exponential mapping with respect to P_F , we can move to $A'B'C'D'$ (see Figs. 2, 3) such that

$$|P_F R| = |\exp_{P_F}^{-1}(R)| = |P_F R'| = w_R,$$

where R belongs to the geodesic $P_F R^\circ$, for $R \in \{A, B, C, D\}$, $R' \in \{A', B', C', D'\}$. Furthermore, we construct the symmetric points of A, C with respect to P_F , A^* and

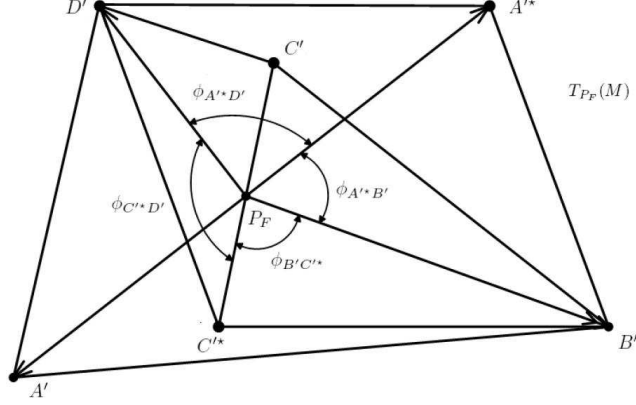


FIGURE 3.

C'^* , respectively, such that $l_{A^*}(P_F) = l_A(P_F)$ and $l_{C^*}(P_F) = l_C(P_F)$ on $T_{P_F}(M)$. Similarly, we construct the symmetric points of A' , C' with respect to P_F , A'^* and C'^* , respectively, such that $l_{A'^*}(P_F) = l_{A'}(P_F)$ and $l_{C'^*}(P_F) = l_{C'}(P_F)$ on $T_{P_F}(M)$ (see Figs. 2 and 3). Taking into consideration Fig. 3, we deduce

$$\begin{aligned} \phi_{B'C'^*} &= \pi - \angle B^\circ P_F C^\circ, & \phi_{A'^*D'} &= \pi - \angle A^\circ P_F D^\circ, \\ \phi_{C'^*D'} &= \pi - \angle C^\circ P_F D^\circ, & \phi_{A'^*B'} &= \pi - \angle A^\circ P_F B^\circ. \end{aligned} \quad (4.19)$$

Replacing (4.19) in (4.17), (4.18), we get $l_{A'^*}(B') = l_{C'^*}(D')$, $l_{A'^*}(D') = l_{B'}(C'^*)$. Therefore, $A'^*B'C'^*D'$ is a parallelogram on $T_{P_F}(M)$. \square

Proposition 3. *If $w_{A^\circ} = w_{C^\circ}$ and $w_{B^\circ} = w_{D^\circ}$, then $A^\circ B^\circ C^\circ D^\circ$ is transformed directly with respect to P_F to a parallelogram $A'B'C'D'$.*

Proof of Proposition 3: Applying the invariance property (geometric plasticity) of P_F we transform the initial quadrilateral $A^\circ B^\circ C^\circ D^\circ$ to $ABCD$ and applying the inverse of the exponential mapping with respect to P_F , we get $A'B'C'D'$ such that

$$|P_F R| = |\exp_{P_F}^{-1}(R)| = |P_F R'| = w_R,$$

for $R \in \{A, B, C, D\}$ and $R' \in \{A', B', C', D'\}$. We conclude that $A'B'C'D'$ is a parallelogram because the diagonals bisect. \square

We proceed by proving the following theorem:

Theorem 3. *The geometrization of the plasticity of convex quadrilaterals on M is given by the following two classes of parallelograms:*

Class A: Parallelograms derived from the initial quadrilateral $A^\circ B^\circ C^\circ D^\circ$ to the tangent plane at P_F by taking the symmetric points of A' and C' with respect to P_F such that

$$|P_F R| = |\exp_{P_F}^{-1}(R)| = |P_F R'| = w_R,$$

where R lies on the geodesic $P_F R^\circ$ for $R \in \{A, B, C, D\}$ and $R' \in \{A', B', C', D'\}$.

Class B: Parallelograms derived from the initial quadrilateral to the tangent plane at P_F by taking the symmetric points of B' and D' with respect to P_F .

Proof of Theorem 3: We consider the following two cases:

1. $w_{B^\circ} > w_{A^\circ} > w_{D^\circ} > w_{C^\circ}$.

Applying Theorem 2, we derive class A of Theorem 3.

2. $w_{A^\circ} > w_{B^\circ} > w_{C^\circ} > w_{D^\circ}$.

Setting $A^\circ \rightarrow B^\circ$, $B^\circ \rightarrow A^\circ$, $C^\circ \rightarrow D^\circ$, $D^\circ \rightarrow C^\circ$, and applying Theorem 2, we derive class B of Theorem 3.

We note that class A and class B contain similar and non similar parallelograms. Similar parallelograms are derived by the condition of the 4-inverse w. F-T problem:

$$w_{A^\circ} + w_{B^\circ} + w_{C^\circ} + w_{D^\circ} = \text{const},$$

by scaling the constant which takes positive real values. \square

Example 2. We consider for simplicity a convex quadrilateral $A^\circ B^\circ C^\circ D^\circ$ in \mathbb{R}^2 , with given distances, angles and weights

$$\begin{aligned} l_{A^\circ} &= 5, \quad l_{B^\circ} = 7.5, \quad l_{C^\circ} = 5, \quad l_{D^\circ} = 10, \\ \angle A^\circ P_F B^\circ &= 120^\circ, \quad \angle B^\circ P_F C^\circ = 90^\circ, \quad \angle C^\circ P_F D^\circ = 50^\circ, \quad \angle D^\circ P_F A^\circ = 100^\circ, \\ w_{A^\circ} &= 0.81, \quad w_{B^\circ} = 0.712, \quad w_{C^\circ} = 0.444, \quad w_{D^\circ} = 0.4 \end{aligned}$$

taken from [9, Example 4.7, p. 418], for the case of \mathbb{R}^2 , where P_F is the corresponding weighted Fermat-Torricelli point which is the intersection of the four prescribed lines $P_F R$ for $R \in \{A^\circ, B^\circ, C^\circ, D^\circ\}$ (see Fig. 4). The convex quadrilateral $A^\circ B^\circ C^\circ D^\circ$ of Fig. 5 has the same angles $\angle R P_F S$, and lengths l_R , for $R, S \in \{A^\circ, B^\circ, C^\circ, D^\circ\}$, like in Fig. 4 with weights $w_{A^\circ} = 0.76, w_{B^\circ} = 0.76, w_{C^\circ} = 0.34, w_{D^\circ} = 0.5$ taken from [9, Example 4.7]. The plasticity equations of Theorem 1 and Corollary 1 show that the w. F-T point P_F remains the same for Figures 4 and 5. We notice that an increase of w_{D° from 0.4 to 0.5 causes a decrease to w_{A° , w_{C° and an increase to w_{B° (plasticity). The weights that correspond to Fig. 4 derive a parallelogram $A'^* B' C'^* D'$ that belongs to class A and the weights that correspond to Fig. 5 derive a parallelogram $A' B' C' D'^*$ that belongs to class B. The weights of Figures 4 and 5 satisfy the plasticity equations for $\text{const} = 2.37$.

Remark 4. We have excluded some trivial cases where the weighted quadrilaterals can be transformed directly to parallelograms and we consider quadrilaterals that the initial starting values of their weights which satisfy the plasticity equations must also satisfy the inequalities $w_{B^\circ} > w_{A^\circ} > w_{D^\circ} > w_{C^\circ}$ or $w_{A^\circ} > w_{B^\circ} > w_{C^\circ} > w_{D^\circ}$.

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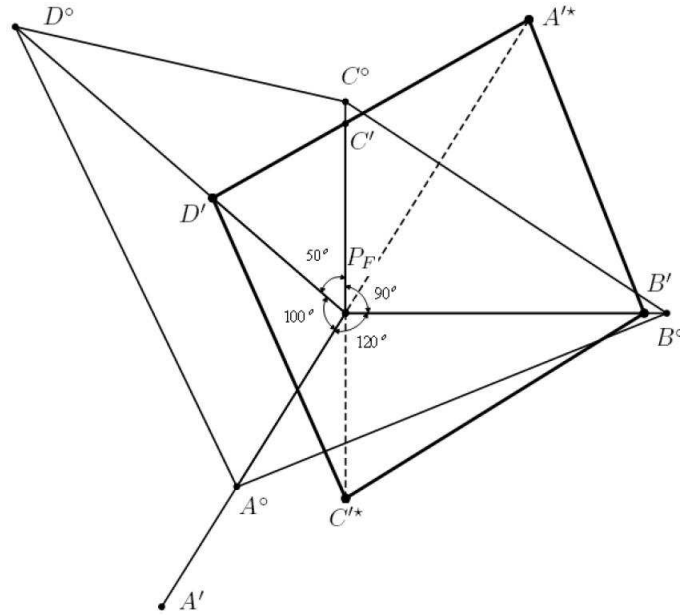


FIGURE 4.

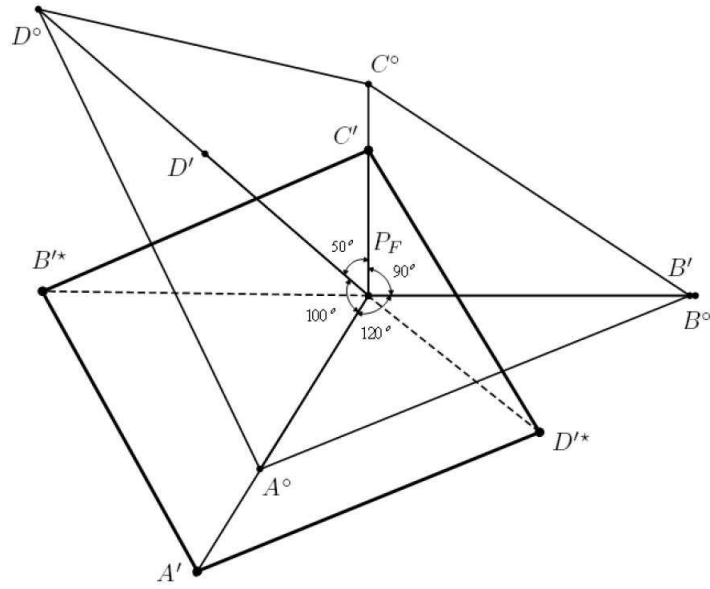


FIGURE 5.

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